# INTERSCHOOL MATHEMATICAL COMPETITION 1984

## Part A

Saturday, 21 July 1984 — 1000-1100

Attempt as many questions as you can. Circle your answers on the Answer sheet provided.

Each question carries 5 marks.

1. In the decimal representation of $13^{62}$, the unit digit is
   (a) 1, (b) 3, (c) 7, (d) 9, (e) none of the preceding.

2. If $\log_{10}x + \log_{10}y = 2$, then the minimum value of $1/x + 1/y$ is
   (a) 1/2, (b) 1/3, (c) 1/4, (d) 1/5, (e) none of the preceding.

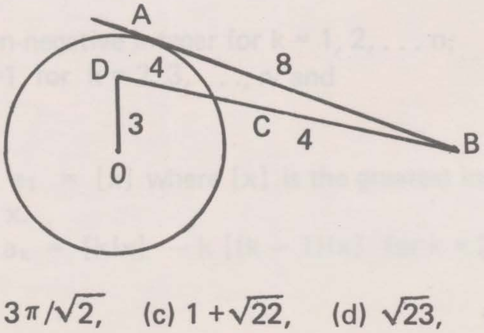
3. If the function f defined by

$$f(x) = \frac{ax}{bx+1}, \quad bx \neq -1,$$

   where a, b are constants, satisfies $f(f(x)) = x$ for all real numbers x for which $bx \neq -1$, then
   (a) $a = 1$, b is arbitrary,
   (b) $b = 0$, a is arbitrary,
   (c) $a = -1$, b is arbitrary,
   (d) $a = b$, b is arbitrary,
   (e) $a = -b$, b is arbitrary.

4. Find the sum of all positive integers less than 10,000 which do not contain any of the three digits 3, 5, and 7.
   (a) 11432190, (b) 11432290, (c) 11432390, (d) 11432490,
   (e) none of the preceding.

5. In the adjoining figure, AB is tangent at A to the circle with centre O; point D is interior to the circle; and DB intersects the circle at C. If BC = DC = 4, OD = 3 and AB = 8, then the radius of the circle is :




   (a) $4+\sqrt{3}$, (b) $3\pi/\sqrt{2}$, (c) $1+\sqrt{22}$, (d) $\sqrt{23}$, (e) $\sqrt{41}$.

6. Straight lines DE, FK and MN parallel to the sides AB, AC and BC respectively of a triangle ABC are drawn through an arbitrary point O lying inside the triangle so that the points F and M are on AB, the points E and K are on BC and the points N and D on AC. Then

$$\frac{AF}{AB} + \frac{BE}{BC} + \frac{CN}{CA}$$

is equal to
(a) 1/3, (b) 1/2, (c) 7/8, (d) 2, (e) none of the preceding.

7. $\sqrt[4]{3+\sqrt{3+\sqrt{2}}}$ is the value of a root of the equation :
(a) $(x^6 - 6x^3 + 6)^2 = 4$, (b) $(x^5 - 6x + 4)^2 = 2$,
(c) $(x^8 + 3x^4 + 8)^2 = 3$, (d) $(x^8 - 3x^4 + 6)^2 = 2$,
(e) $(x^8 - 6x^4 + 6)^2 = 2$.

8. The largest integral value of n such that there exists a solution to the system of inequalities

$$k < x^k < k+1, \ k = 1, 2, \ldots, n$$

is:
(a) 2, (b) 3, (c) 4, (d) 5, (e) none of the preceding.

9. An integer is randomly chosen from 1000 to 1000000 (inclusively). The probability that the integer chosen is neither the square nor the cube of an integer is:
(a) 994798/999001, (b) 994536/999999, (c) 979748/999099,
(d) 997948/999001, (e) 997945/999011.

10. Three balanced dice are thrown and the scores $N_1$, $N_2$ and $N_3$ are noted. Then $N_1$ white balls, $N_2$ black balls and $N_3$ red balls are placed in an empty urn. A ball is drawn from the urn. Find the probability that the ball drawn is red.
(a) 1/216, (b) 1/3, (c) 1/2, (d) 1/6, (e) none of the preceding.

–THE END–

Interschool Mathematical Competition 1984

PART B

Saturday, 21 July 1984 1100 – 1300

Attempt as many questions as you can.

Each question carries 25 marks.

1. Grace, Helen and Mary were discussing their ages one day, and in the course of their conversation they made the following assertions:

   Grace : I am twenty-two years old.
   I am two years younger than Helen.
   I am a year older than Mary.

   Helen : I am not the youngest.
   Mary and I are three years apart.
   Mary is twenty-five years old.

   Mary : I am younger than Grace.
   Grace is twenty-three years old.
   Helen is three years older than Grace.

   It is of course too much to expect that three young women should be entirely truthful when speaking of their ages, and in the present instance only two of the three statements made by each girl are true. Can you deduce the age of each one?

2. Prove that for any integer $n \geqslant 3$, $(n!)^2 > n^n$.

3. Show that the only solutions in positive integers x,y of the equation :
$$3 \cdot 2^x + 1 = y^2 \text{ are } (x,y) = (3,5),\ (4,7).$$

4. Let x be a positive rational number. Suppose that
$$x = \sum_{k=1}^{n} \frac{a_k}{k!} \quad \text{where}$$

   (i) $a_k$ is a non-negative integer for $k = 1, 2, \ldots n$;
   (ii) $a_k \leq k-1$ for $k = 2, 3, \ldots, n$ and
   (iii) $a_n > 0$.

   Show that (a) $a_1 = [x]$ where $[x]$ is the greatest integer less than or equal to x.
   (b) $a_k = [k!x] - k\,[(k-1)!x]$ for $k = 2, 3, \ldots, n$.

5. In City S, streets 1 to $m + 1$ are one-way streets running northwards, while avenues 1 to $n$ allow one-way traffic going eastwards between streets 1 and $m$ and two-way traffic between streets $m$ and $m + 1$. A man wishes to drive from the junction of street 1 and avenue 1 to the junction of street $m$ and avenue $n$. How many different routes can he take if he is allowed to travel on street $m + 1$ at most once on his trip?

6. (i) Let $a_1, \ldots, a_n$ be positive numbers. Prove that

$$\sum_{(i_1,\ldots,i_n)} \frac{1}{a_{i_1}(a_{i_1}+a_{i_2})\ldots(a_{i_1}+\ldots+a_{i_n})} = \frac{1}{a_1\ldots a_n}$$

where the sum is taken over all permutations $(i_1,\ldots i_n)$ of $(1,\ldots,n)$

(ii) Ten cards numbered 1 to 10 are shuffled and placed face down. One card is turned up, its number $N_1$ is noted and it is discarded. $N_1$ balls of the same colour are placed in an empty urn. One ball is then drawn from the urn and replaced after noting its colour. Next, a second card is turned up, its number $N_2$ is noted and it is discarded. $N_2$ balls of a colour different from those in the urn are placed in the urn. A ball is then drawn from the urn and replaced after noting its colour. This process is repeated until all the 10 cards are turned up. Use part (i) or any other method to prove that the probability that the colours of the 10 balls drawn are all distinct is equal to 1/10!.

– THE END –